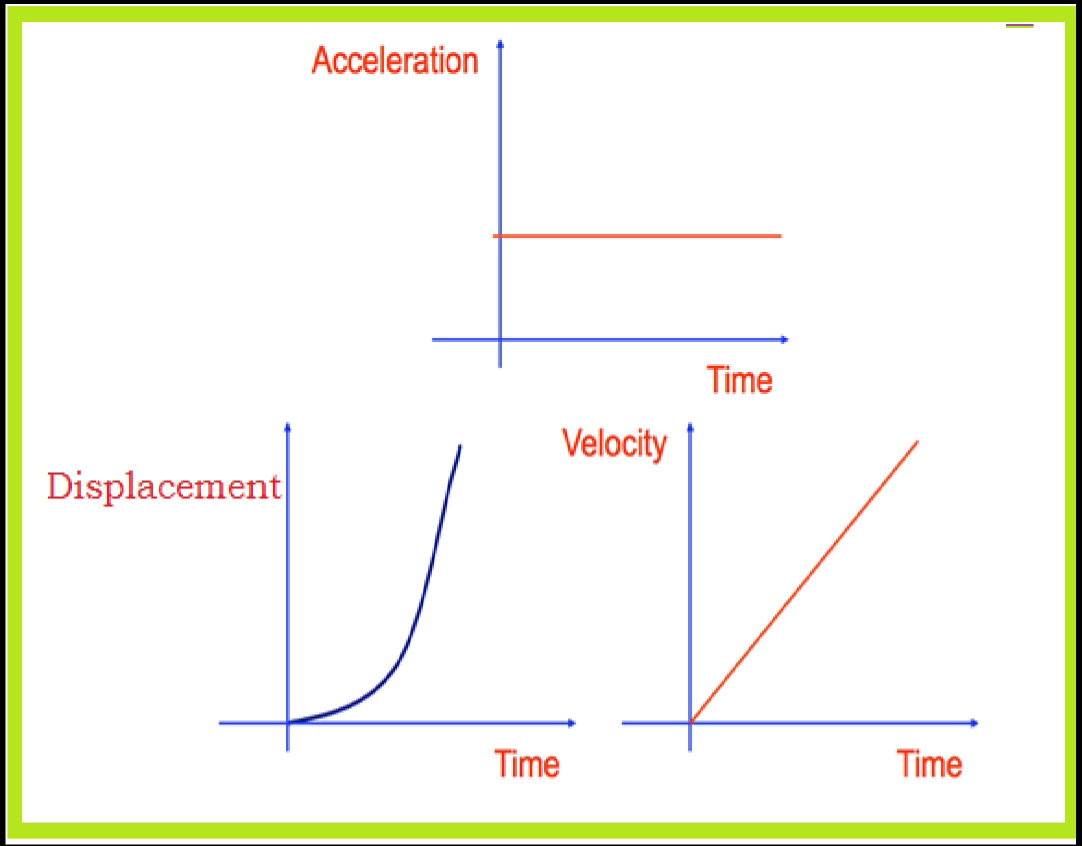
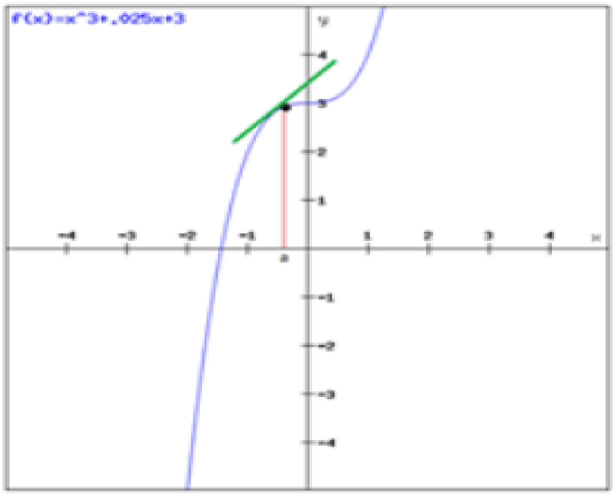
SECOND ORDER DERIVATIVE
MODULE-7



$y = f(x): \frac{dy}{dx} = f'(x) : \frac{d^2y}{dx^2} = f''(x)$



In this graph the blue line indicates the slope i.e. the first derivative of the given function. And the second derivative is used to define the nature of the given function. For example, we use the second derivative test to determine the maximum, minimum or the point of inflexion.

Mathematically, if $y = f(x)$

Then $\frac{uy}{dx}$ = f'(x)

Now if f'(x) is differentiable, then differentiating $\frac{dy}{dx}$ again w.r.t. x we get 2nd order derivative, i.e.

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ = f"(x)

Similarly, higher order derivatives can also be defined in the same fashion like $\frac{d^3y}{dx^3}$ represents a third order derivative $\frac{d^4y}{dx^4}$, represents a fourth order derivative and so on.

The Second Derivative

The second derivative is what you get when you differentiate the derivative. Remember that the derivative of y with respect to x is written dy/dx. The second derivative is written d^2y/dx^2, pronounced "dee two y by d x squared".

If $y = x^3$	$f(x) = x^3$
$\frac{dy}{dx} = 3x^2$	$f'(x) = 3x^2$
$\frac{d^2y}{dx^2} = 6x$	$f''(x) = 6x$

Two alternative ways of writing the second derivative

Find the second order derivatives of the function.

$\sin(\log x)$

Solution 2

f(g(x))

Let $y = \sin(\log x)$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}[\sin(\log x)] = \cos(\log x) \cdot \frac{d}{dx}(\log x) = \frac{\cos(\log x)}{x}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right]$$

$$= \frac{x \cdot \frac{d}{dx}[\cos(\log x)] - \cos(\log x) \cdot \frac{d}{dx}(x)}{x^2}$$

$$= \frac{-x\sin(\log x) \cdot \frac{1}{x} - \cos(\log x)}{x^2}$$

$$= \frac{-[\sin(\log x) + \cos(\log x)]}{x^2}$$

Question

Find the second order derivatives of the function.

$e^{6x}\cos 3x$

Let $y = e^{6x}\cos 3x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{6x}\cdot\cos 3x\right) = \cos 3x\cdot\frac{d}{dx}\left(e^{6x}\right) + e^{6x}\cdot\frac{d}{dx}(\cos 3x)$$

$$= \cos 3x\cdot e^{6x}\cdot\frac{d}{dx}(6x) + e^{6x}\cdot(-\sin 3x)\cdot\frac{d}{dx}(3x)$$

$$= 6e^{6x}\cos 3x - 3e^{6x}\sin 3x \qquad \ldots(1)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(6e^{6x}\cos 3x - 3e^{6x}\sin 3x\right) = 6\cdot\frac{d}{dx}\left(e^{6x}\cos 3x\right) - 3\cdot\frac{d}{dx}\left(e^{6x}\sin 3x\right)$$

$$= 6\cdot\left[6e^{6x}\cos 3x - 3e^{6x}\sin 3x\right] - 3\cdot\left[\sin 3x\cdot\frac{d}{dx}\left(e^{6x}\right) + e^{6x}\cdot\frac{d}{dx}(\sin 3x)\right]$$

$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 3\left[\sin 3x\cdot e^{6x}\cdot 6 + e^{6x}\cdot\cos 3x\cdot 3\right]$$

$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 18e^{6x}\sin 3x - 9e^{6x}\cos 3x$$

$$= 27e^{6x}\cos 3x - 36e^{6x}\sin 3x$$

$$= 9e^{6x}\left(3\cos 3x - 4\sin 3x\right)$$

Question

If $y = e^{-x}\cos x$, show that $\frac{d^2y}{dx^2} = 2e^{-x}\sin x$

Solution

$$y = e^{-x}\cos x$$

differentiating both sides w.r.t x

$$\Rightarrow \quad \frac{dy}{dx} = e^{-x}(-\sin x) + (\cos x)(-e^{-x})$$

$$\Rightarrow \quad \frac{dy}{dx} = -e^{-x}\sin x - e^{-x}\cos x = -e^{-x}(\sin x + \cos x)$$

again differentiating both sides w.r.t. x

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -e^{-x}(\cos x - \sin x) + e^{-x}(\sin x + \cos x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 2e^{-x}\sin x$$

EXPONENTIAL & TRIGONOMETRIC FUNCTIONS

Y = uv

Question

If $y = 2\sin x + 3\cos x$, show that $\frac{d^2y}{dx^2} + y = 0$

Solution

$y = 2\sin x + 3\cos x$

differentiating w.r.t. x

$$\Rightarrow \quad \frac{dy}{dx} = 2\cos x + 3(-\sin x) = 2\cos x - 3\sin x$$

differentiating w.r.t. x

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 2(-\sin x) - 3\cos x = -(2\sin x + 3\cos x) = -y$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + y = 0$$

KEEP ON CHANGING

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NEVER DISAPPEAR

Question

If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\dfrac{d^2y}{dx^2}$

Solution

It is given that, $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

$$\therefore \frac{dx}{dt} = a\cdot\frac{d}{dt}(\cos t + t\sin t)$$

$$= a\left[-\sin t + \sin t\cdot\frac{d}{dt}(t) + t\cdot\frac{d}{dt}(\sin t)\right]$$

$$= a[-\sin t + \sin t + t\cos t] = at\cos t$$

$$\frac{dy}{dt} = a\cdot\frac{d}{dt}(\sin t - t\cos t)$$

$$= a\left[\cos t - \left\{\cos t\cdot\frac{d}{dt}(t) + t\cdot\frac{d}{dt}(\cos t)\right\}\right]$$

$$= a[\cos t - \{\cos t - t\sin t\}] = at\sin t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{at\sin t}{at\cos t} = \tan t$$

Then, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}(\tan t) = \sec^2 t\cdot\dfrac{dt}{dx}$

$$= \sec^2 t\cdot\frac{1}{at\cos t} \qquad \left[\frac{dx}{dt} = at\cos t \Rightarrow \frac{dt}{dx} = \frac{1}{at\cos t}\right]$$

$$= \frac{\sec^3 t}{at}$$

CAREFULL

PARAMETRIC FUNCTION WITH SECOND DERIVATIVES

Question

If $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$, find $\dfrac{d^2y}{dx^2}$

Solution

$x = a(\theta - \sin\theta); \; y = a(1 + \cos\theta)$

Diiferentiating the above functions with respect to θ, we get,

$$\frac{dx}{d\theta} = a(1 - \cos\theta) \quad ...(1)$$

$$\frac{dy}{d\theta} = a(-\sin\theta) \quad ...(2)$$

Dividing equation (2) by (1), we have,

$$\frac{dy}{dx} = \frac{a(-\sin\theta)}{a(1-\cos\theta)} = \frac{-\sin\theta}{1-\cos\theta}$$

Differentiate again w.r.t x

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{(1-\cos\theta)(-\cos\theta) + \sin\theta(\sin\theta)}{(1-\cos\theta)^2} \times \frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{-\cos\theta + \cos^2\theta + \sin^2\theta}{(1-\cos\theta)^2} \cdot \times \frac{1}{a(1-\cos\theta)}$$

$$= \frac{1-\cos\theta}{(1-\cos\theta)^2} \times \frac{1}{a(1-\cos\theta)}$$

$$= \frac{1}{a\,(1-\cos\theta)^2}$$

$$= \frac{1}{a\left(2\sin^2\frac{\theta}{2}\right)^2}$$

$$= \frac{1}{4a\sin^4\left(\frac{\theta}{2}\right)}$$

$$= \frac{1}{4a}\operatorname{cosec}^4\left(\frac{\theta}{2}\right)$$

Question

If $y = (\sin^{-1} x)^2$ prove that $(1-x^2)y_2 - xy_1 - 2 = 0$

$y = (\sin^{-1} x)^2$

differentiating w.r.t. x

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2\sin^{-1} x}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,\frac{dy}{dx} = 2\sin^{-1} x$$

Squaring both sides

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = 4\,(\sin^{-1} x)^2$$

Differentiating w.r.t

$$(1-x^2)2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2(-2x) = 4.2.\sin^{-1} x\,\frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx}\left\{2(1-x^2)\frac{d^2y}{dx^2} + (-2x)\left(\frac{dy}{dx}\right)\right\} = 4.\ \frac{2.\sin^{-1} x}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx}\left\{2(1-x^2)\frac{d^2y}{dx^2} + (-2x)\left(\frac{dy}{dx}\right)\right\} = 4.\ \frac{dy}{dx}$$

$$2\left\{(1-x^2)\frac{d^2y}{dx^2} + (-x)\left(\frac{dy}{dx}\right)\right\} = 4.\ 2$$

$$(1-x^2)\,y_2 - xy_1 = 2$$

$$(1-x^2)y_2 - xy_1 - 2 = 0$$

A very important Question